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Winter Term 2024/25
06.02.2025

Algebraic number theory

Exam

Family Name: ______________________ First name: ______________________

Student ID: ______________________ Term: ______________________

Degree course: Bachelor, PO ❑ 2011 ❑ 2015 ❑ 2021 Master, PO ❑ 2011 ❑ 2021

Lehramt Gymnasium: ❑ modularisiert ❑ nicht modularisiert

❑ Diplom ❑ Other: ______________________

Major subject: ❑ Mathematik ❑ Wirtschaftsm. ❑ Inf. ❑ Phys. ❑ Stat. ❑ ______

Minor subject: ❑ Mathematik ❑ Wirtschaftsm. ❑ Inf. ❑ Phys. ❑ Stat. ❑ ______

Credit Points to be used for ❑ Hauptfach ❑ Nebenfach (Bachelor / Master)

Please switch off your mobile phone and do not place it on the table; place your identity and student ID cards on the table so that they are clearly visible.
Please check that you have received all **four problems**.
Please do not write with the colours red or green. Write **on every page** your **family name and your first name**.
Write your solutions on the page marked with the appropriate problem number. If you run out of space, use the empty pages at the end of the examination paper ensuring that each problem is clearly marked.
Please make sure to submit only one solution for each problem; cross out everything that should not be graded.

Good luck!

1	2	3	4	$\sum$
/8	/21	/20	/24	/73

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Problem 1. **[3+3 Points]**

1) Let $K = \mathbb{Q}(\sqrt{3}, \sqrt{7})$ and $\alpha = \frac{\sqrt{3}+\sqrt{7}}{2} \in K$. Show that α is integral over $\mathbb{Z}$.
Hint: Show that α is a root of the polynomial $X^2 - \sqrt{3}X - 1$.

2) Let $L = \mathbb{Q}(\sqrt{3}, \sqrt{5})$ and $\alpha = \frac{\sqrt{3}+\sqrt{5}}{2} \in L$. Show that α is not integral over $\mathbb{Z}$.
Hint: Compute $N^L_{\mathbb{Q}}(\alpha)$. Recall that $L/\mathbb{Q}$ is a Galois extension of degree 4 and $\mathrm{Gal}(L/\mathbb{Q}) = \{id, \sigma_1, \sigma_2, \sigma_1\sigma_2\}$, where id is the identity map, $\sigma_1(\sqrt{3}) = -\sqrt{3}$, $\sigma_1(\sqrt{5}) = \sqrt{5}$ and $\sigma_2(\sqrt{3}) = \sqrt{3}$, $\sigma_2(\sqrt{5}) = -\sqrt{5}$.

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Problem 2. **[2+4+2+2+3+2+4 = 19 Points]**

One can check that the polynomial $X^3 - X - 1$ has two complex and one real root $\alpha \in \mathbb{R}$. Let $K = \mathbb{Q}(\alpha)$.

1) Show that $X^3 - X - 1$ is irreducible over $\mathbb{Q}$ and that $[K : \mathbb{Q}] = 3$.

2) Compute the traces $\mathrm{Tr}_{\mathbb{Q}}^K$ of α, α^2, α^3 and α^4.
Hint: Recall that $\mathrm{Tr}_{\mathbb{Q}}^K : K \to \mathbb{Q}$ is a $\mathbb{Q}$-linear map.

3) Show that $D_{\mathbb{Q}}^K(1, \alpha, \alpha^2) = -23$.

4) Show that the absolute discriminant d_K equals -23 and that $1, \alpha, \alpha^2$ is a $\mathbb{Z}$-basis of $\mathcal{O}_K$.

5) Let ρ be a prime ideal in $\mathcal{O}_K$ containing $23\mathcal{O}_K$. Show that the residual index of ρ equals 1.
Hint: Use 4).

6) Show that $\alpha + 1$ is a unit in $\mathcal{O}_K$.

7) Let u be a unit in $\mathcal{O}_K$, such that $u \neq \pm 1$. Show that there exist non-zero integers n and m, such that $(\alpha + 1)^n = u^m$.

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Problem 3. **[4+4+1+2+6=17 Points]**

Let $K = \mathbb{Q}(\sqrt{10})$ be a quadratic field extension of $\mathbb{Q}$.

1) Explicitly compute the decomposition of $2\mathcal{O}_K$ and $3\mathcal{O}_K$ into a product of prime ideals.

2) Deduce from 1) that there exists a unique prime ideal ρ in $\mathcal{O}_K$ with $N(\rho) = 2$. Show that ρ is not a principle ideal.

3) Show that the order of ρ in the class group $C(\mathcal{O}_K)$ is 2.

4) Find r_2 and d_K for the extension $K/\mathbb{Q}$ and show that

$$\left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n} \sqrt{|d_K|} < 4\,,$$

where $n = [K : \mathbb{Q}]$.

5) Deduce from 1) - 4) that $C(\mathcal{O}_K) \simeq \mathbb{Z}/2\mathbb{Z}$.
Hint: To establish a relation between the classes of ideals in $C(\mathcal{O}_K)$ compute the norm $N_{\mathbb{Q}}^{K}(2+\sqrt{10})$ and consider the decomposition of the principle ideal $(2+\sqrt{10})\mathcal{O}_K$ into a product of prime ideals.

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Problem 4. **[6+3+3+6=18 Points]**

1) Let $\mathbb{Q} \subset K$ be a number field and $A \subset K$ a dvr with field of fractions K, maximal ideal $\mathfrak{m}$ and uniformizing element π. Show that $\mathcal{O}_K \subset A$.
Hint: Let $x \in K$. Recall that there is a unique decomposition $x = u\pi^n$, where $u \in A^*$ and n is an integer. Observe that if $x \notin A$, then $x^{-1} \in A$.

2) Show that $\rho := \mathfrak{m} \cap \mathcal{O}_K$ is a non-zero prime ideal in $\mathcal{O}_K$.
Hint: Recall that K is the field of fractions of $\mathcal{O}_K$.

3) Let $(\mathcal{O}_K)_\rho \subset K$ be the localization of $\mathcal{O}_K$ at ρ (that is $(\mathcal{O}_K)_\rho = S^{-1}\mathcal{O}_K$, where $S = \mathcal{O}_K \backslash \rho$). Show that $(\mathcal{O}_K)_\rho \subset A$.

4) Conclude that $A = (\mathcal{O}_K)_\rho$.
Hint: Observe that both $(\mathcal{O}_K)_\rho$ and A are dvrs with K as field of fractions.

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